ERGODIC AND MIXING PROPERTIES OF
EQUILIBRIUM MEASURES FOR MARKOV PROCESSES

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ABSTRACT. Let $S(t)$ be the semigroup corresponding to a Markov process on a metric space X. Suppose $S(t)$ commutes with a homeomorphism T of X. We prove that under certain conditions, an equilibrium measure for the process is ergodic under T. We also show that, under stronger conditions this measure must be mixing under T. Several applications of these results are given.

1. INTRODUCTION AND STATEMENT OF RESULTS

This paper was motivated by two questions raised in the area of interacting particle systems by Liggett [12]. These questions ask whether or not certain equilibrium measures of translation invariant particle systems are ergodic. Our main result answers these questions affirmatively, but, since its proof is applicable to other systems we will state it in a more general context. To do so we start giving the necessary definitions.

Let X be a metric space and $C = C(X)$ the Banach space of all bounded continuous real-valued functions on X, with the following norm:

$$\|f\| = \sup_{\eta \in X} |f(\eta)|.$$

Denote by $\mathscr{M}$ the set of all Borel probability measures on X. $\mathscr{M}$ will be endowed with the topology of weak convergence and its Borel σ-algebra. A homeomorphism T on X induces transformations T^* and T^{**} on C and $\mathscr{M}$ respectively, through the following equalities:

$$(T^* f)(\eta) = f(T\eta) \quad \forall f \in C, \forall \eta \in X,$$

and

$$\int f \, d(\mu T^{**}) = \int (T^* f) \, d\mu \quad \forall f \in C, \forall \mu \in \mathscr{M}.$$

In the sequel we will drop the superscripts of T^* and T^{**} because no confusion will occur.

Now, let $\{T_i\}_{i \in I}$ be a collection of homeomorphisms of X, indexed by a finite set I, such that

$$(1.1) \qquad\qquad T_i T_j = T_j T_i \quad \forall i, j \in I.$$

By a Feller semigroup $\{S(t), t \geq 0\}$ on C we mean

(a) $\forall t \geq 0$, $S(t)$ is a linear operator on C,
(b) $S(0)$ is the identity on C,
(c) $S(t + s) = S(t)S(s)$ $\forall t, s \geq 0$,
(d) $S(t)1 = 1$ $\forall t \geq 0$ where 1 is the function $f(\eta) \equiv 1$,
(e) $S(t)f \geq 0$ $\forall f \geq 0$ in C and all $t \geq 0$,
(f) $\forall t \geq 0$ $\forall \mu \in \mathcal{M}$ there exists $\mu S(t) \in \mathcal{M}$ such that

$$\int f \, d\mu S(t) = \int S(t) f \, d\mu \quad \forall f \in C.$$

Note that this is not the usual definition of a Feller semigroup, since we do not require X to be locally compact and $S(t)$ to be strongly continuous (see page 166 of [6]). It will prove, however, to be convenient for our purposes. Note also that it follows from (a), (d) and (e) that for all $t \geq 0$, $S(t)$ is a continuous operator of norm 1. Finally observe that the uniqueness of the measure $\mu S(t)$ in (f) follows from Proposition 4.2 on page 111 of [6] and the Dominated Convergence Theorem. The existence of $\mu S(t)$ is a consequence of the Riesz Representation Theorem and the continuity of $S(t)$, if X is compact. In view of the uniqueness of $\mu S(t)$ and of c) and f), we may also consider $S(t)$ as a semigroup acting on $\mathcal{M}$ (i.e. $\mu S(t + s) = \mu S(t)S(s)$). Since $S(t)f \in C$ for all $f \in C$, $S(t)$ acts continuously on $\mathcal{M}$ too.

In this paper we will consider Feller semigroups $S(t)$ satisfying

(1.2) $S(t)T_i = T_i S(t)$ $\quad \forall i \in I, \forall t \geq 0.$

The subsets of $\mathcal{M}$ given by $\{\mu : \mu T_i = \mu$ $\forall i \in I\}$ and $\{\mu : \mu S(t) = \mu$ $\forall t \geq 0\}$ will be denoted by $\mathcal{S}$ and $\mathcal{I}$ respectively. Besides this, if $\mathcal{A} \subset \mathcal{M}$, then $\mathcal{A}_e$ will be the set of extreme points of $\mathcal{A}$.

An element $\mu \in \mathcal{S}$ will be said to be T_I-ergodic if any Borel measurable real-valued function f on X, such that $T_i f = f$ μ-a.e. $\forall i \in I$, is μ-a.e. constant.

Since we will deal with multi-indexed sequences, it is convenient to introduce the following notation: k_I and n_I will be sequences of nonnegative integers indexed by I (i.e $k_I = (k_i)_{i \in I}$), expressions as $k_I \leq n_I$ or $n_I \to \infty$ will be interpreted componentwise and $|n_I|$, $\prod n_I$ and T^{k_I} will denote $\sum_{i \in I} n_i$, $\prod_{i \in I}(n_i + 1)$ and $\prod_{i \in I} T_i^{k_i}$ respectively.

If $\mu \in \mathcal{S}$, then T_i can be extended to $L^2(\mu)$ by means of $(T_i f)(\eta) = f(T_i \eta)$ $\forall \eta \in X$, $\forall f \in L^2$, and it is well known that the following three conditions are equivalent:

(1) μ is T_I-ergodic,
(2) $\lim_{n_I \to \infty}(1/\prod n_I) \sum_{k_I \leq n_I} \int f(T^{k_I} g) \, d\mu = \int f \, d\mu \cdot \int g \, d\mu$ $\forall f, g \in L^2(\mu)$,
(3) $\mu \in \mathcal{S}_e$.

For a proof of these equivalences the reader is referred to Chapter I of [12].

The proofs of Proposition 4.11 and Corollary 4.14 in that chapter can be easily adapted to the present situation.

Let $\nu \in \mathscr{S}$ and suppose there exists a Borel probability measure λ on $\mathscr{S}_e$, such that

$$(1.3) \qquad \nu = \int_{\mathscr{S}_e} \mu \, d\lambda(\mu).$$

This is called an ergodic decomposition of ν. The existence (and uniqueness) of such a decomposition for all elements of $\mathscr{S}$ follows from Theorems 3.1 and 6.1 of [5], if X is a complete separable metric space.

An element $\mu \in \mathscr{S}$ is called T_I-mixing if

$$\lim_{|n_I| \to \infty} \int f(T^{n_I} g) \, d\mu = \int f \, d\mu \int g \, d\mu \quad \forall f, g \in L^2(\mu),$$

where the T_i's have been extended as before to $L^2(\mu)$. Obviously if μ is T_I-mixing, then it is T_I-ergodic.

Suppose $S(t)$ and $\{T_i\}$ are as above and μ is an element of $\mathscr{I} \cap \mathscr{S}$. When is it the case that μ is T_I-ergodic? When is it T_I-mixing? Theorem (1.4) below gives sufficient conditions for the answers to be affirmative. In its statement, as well as in the sequel, δ_η will denote the Dirac measure on η.

Theorem (1.4). *Let X be a metric space, $S(t)$ a Feller semigroup on $C(X)$ and $\{T_i\}_{i \in I}$ a finite collection of homeomorphisms on X satisfying (1.1) and (1.2). Suppose $\nu \in \mathscr{I} \cap \mathscr{S}$ and B is a subset of $C(X)$ which is dense in $L^2(\nu)$. Moreover assume that*

$$(1.5) \qquad \lim_{|n_I| \to \infty} \int |S(t)[fT^{n_I}g] - S(t)fS(t)(T^{n_I}g)| \, d\nu = 0 \quad \forall f, g \in B.$$

(i) *If $\nu = \int \mu \, d\lambda(\mu)$ is an ergodic decomposition of ν and there exists a sequence $t_n \uparrow \infty$, such that $\lim_n \int f \, d\mu S(t_n) = \int f \, d\nu \; \forall f \in B$ and λ-almost all μ, then ν is T_I-ergodic.*

(ii) *If there exists a sequence $t_n \uparrow \infty$, such that $\lim_n \int f d(\delta_\eta S(t_n)) = \int f \, d\nu$ $\forall f \in B$ and ν-almost all η, then ν is T_I-mixing.*

In §2 we prove Theorem (1.4). §§3–6 are applications of that theorem. These applications can be read in any order, except for the fact that Proposition (4.1) is needed in the first part of §5. §7 contains examples to show that one cannot drop any of the hypotheses of Theorem (1.4). Some open problems are also stated in §7. In §§3–7, X will be a translation invariant subspace of $W^{\mathbf{Z}^d}$, where W is a metric space, $\mathbf{Z}^d$ is the d-dimensional integer lattice and the T_i's will be the operators on X induced by the translations on $\mathbf{Z}^d$. Since the

T_i's are fixed in those sections, we will write that an element μ of $\mathscr{S}$ is ergodic or mixing instead of T_I-ergodic or T_I-mixing respectively.

2. Proof of Theorem (1.4)

Let f, $g \in B$ and $\mu \in \mathscr{M}$. Then define

$$A(n_I, f, g, \mu) = \int f T^{N_I} g \, d\mu$$

and

$$B(n_I, f, g, \mu) = \frac{1}{\prod n_I} \sum_{k_I \le n_I} A(k_I, f, g, \mu).$$

To prove part (i) write

$$B(n_I, f, g, \nu) = B(n_I, f, g, \nu S(t))$$
$$= \frac{1}{\prod n_I} \sum_{k_I \le n_I} \int \left[\int f T^{k_I} g \, d\mu S(t) \right] d\lambda(\mu).$$

Since

$$\int f T^{k_I} g \, d\mu S(t) = \int S(t) \left[f T^{k_I} g \right] d\mu$$
$$= \int S(t) f T^{k_I} [S(t)g] \, d\mu$$
$$+ \int \left(S(t) [f T^{k_I} g] - S(t) f T^{k_I} (S(t)g) \right) d\mu,$$

we obtain the following equality:

$$B(n_I, f, g, \nu) = \int \left[\frac{1}{\prod n_I} \sum_{k_I \le n_I} \int S(t) f T^{k_I} (S(t)g) \, d\mu \right] d\lambda(\mu)$$
$$+ \frac{1}{\prod n_I} \sum_{k_I \le n_I} \int \left(S(t) [f T^{k_I} g] - S(t) f T^{k_I} (S(t)g) \right) d\nu.$$

It now follows from (1.5) and the fact that μ is T_I-ergodic for λ-almost all μ, that

$$(2.1) \quad \lim_{n_I \to \infty} B(n_I, f, g, \nu) = \int \left[\int S(t) f \, d\mu \cdot \int S(t) g \, d\mu \right] d\lambda(\mu) \quad \forall t \ge 0.$$

Taking limits in (2.1) along the sequence t_n given by part (i) of Theorem (1.4) and using the Dominated Convergence Theorem we get

$$\lim_{n_I} B(n_I, f, g, \nu) = \int f \, d\nu \int g \, d\nu \quad \forall f, g \in B.$$

Since B is dense in $L^2(\nu)$, the same equality holds for all $f, g \in L^2(\nu)$. Hence, ν is T_I-ergodic.

For (ii) write for $t \geq 0$

$$(2.2) \quad A(n_I, f, g, \nu) = A(n_I, f, g, \nu S(t))$$

$$= \int \left[S(t)(fT^{n_I}g)(\eta) - S(t)f(\eta)S(t)(T^{n_I}g)(\eta) \right] d\nu(\eta)$$

$$+ \int S(t)f(\eta) \left[S(t)(T^{n_I}g)(\eta) - \int g(\xi) \, d\nu(\xi) \right] d\nu(\eta)$$

$$+ \int S(t)f(\eta) \, d\nu(\eta) \cdot \int g(\xi) \, d\nu(\xi).$$

The absolute value of the second term of the right-hand side is bounded above by

$$\|f\| \int \left| S(t)(T^{n_I}g)(\eta) - \int g(\xi) \, d\nu(\xi) \right| d\nu(\eta).$$

It now follows from the invariance of ν and $S(t)$ under T^{n_I} that this bound is equal to

$$\|f\| \int \left| S(t)g(\eta) - \int g(\xi) \, d\nu(\xi) \right| d\nu(\eta).$$

Note that this bound is independent of η_I and that it converges to 0 if t goes to infinity along the sequence t_n given in part (ii) of Theorem (1.4). Therefore, given $\varepsilon > 0$ there exists an integer n_0 such that for all $n \geq n_0$

$$\limsup_{n_I \to \infty} A(n_I, f, g, \nu)$$

$$\leq \limsup_{n_I \to \infty} \int [S(t_n)(fT^{n_I}g)(\eta) - S(t_n)f(\eta)S(t_n)(T^{n_I}g)(\eta)] \, d\nu(\eta)$$

$$+ \int S(t_n)f \, d\nu(\eta) \int g(\xi) \, d\nu(\xi) + \varepsilon.$$

Using (1.5) we see that the first term of the right-hand side above is 0. Hence,

$$\limsup_{n_I \to \infty} A(n_I, f, g, \nu) \leq \lim_n \int S(t_n)f \, d\nu(\eta) \cdot \int g(\xi) \, d\nu(\xi) + \varepsilon$$

$$= \int f \, d\nu \cdot \int g \, d\nu + \varepsilon.$$

A similar argument shows that

$$\liminf_{n_I \to \infty} A(n_I, f, g, \nu) \geq \int f \, d\nu \int g \, d\nu - \varepsilon.$$

Since ε is arbitrary we have $\forall f, g \in B$

$$\lim_{n_I \to \infty} A(n_I, f, g, \nu) = \int f \, d\nu \int g \, d\nu.$$

Using the fact that B is dense in $L^2(\nu)$, we see that the same equality holds for all $f, g \in L^2(\nu)$. This completes the proof of part (ii).

3. ERGODIC INTERACTING PARTICLE SYSTEMS

In this section we consider the Markov processes constructed in §I-3 of [12]. These processes have as state space $X = W^S$, where W is a compact metric

space and S is a finite or countable set. The space X, with the product topology, becomes a compact metric space. To describe the processes we quote [12]: "The local dynamics of the system are described by a collection of transition measures $c_T(\eta, d\zeta)$. For each $\eta \in X$ and finite $T \subset S$, $c_T(\eta, d\zeta)$ is assumed to be a finite positive measure on W^T. We will assume further that for each T the mapping $\eta \to c_T(\eta, d\zeta)$ is continuous from X to the space of finite measures on W^T, with the topology of weak convergence. The interpretation is that η is the current configuration of the system, $c_T(\eta, W^T)$ is the rate at which a transition will occur involving the coordinates in T, and $c_T(\eta, d\zeta)/c_T(\eta, W^T)$ is the distribution of the restriction to T of the new configuration after that transition has taken place." In this application S will be the d-dimensional integer lattice $\mathbf{Z}^d$ and the local dynamics of the processes will be invariant under translations on $\mathbf{Z}^d$. Then, as shown in [12], under the assumptions

$$(3.1) \qquad \sum_{K \ni 0} \sup_{\eta \in X} \{c_K(\eta, W^K)\} < \infty$$

and

$$(3.2)$$
$$\sum_{K \ni 0} \sum_{u \neq 0} \sup\{\|c_K(\eta_1, d\xi) - c_K(\eta_2, d\xi)\|_K : \eta_1(y) = \eta_2(y) \text{ for all } y \neq u\} < \infty$$

($\|\cdot\|_K$ being the total variation norm of a measure on W^K), there exists a Feller semigroup $\{S(t), t \geq 0\}$, acting on $C(X)$ whose generator corresponds to the dynamics described above. Since this generator is invariant under translations, it follows from part b) by Theorem 2.9 of Chapter I of [12] that the semigroup $S(t)$ shares this property. Moreover, under the further assumption

$$(3.3) \qquad \sum_{K \ni 0} |K| \sup_{n \in X} \{C_K(\eta, W^K)\} < \infty$$

($|\cdot|$ denoting the cardinality of a set), it follows from Theorem 4.6 on pages 35–36 of [12] that (1.5) holds for all continuous f and g depending on finitely many coordinates.

An interacting particle system on X is called ergodic if there exists a (necessarily unique) measure $\nu \in \mathcal{M}$ such that $\lim_{t \to \infty} \mu S(t) = \nu$ for all $\mu \in \mathcal{M}$. From these definitions, the above considerations and part (ii) of Theorem (1.4) we conclude that:

If a translation invariant particle system on $W^{\mathbf{Z}^d}$ satisfies (3.3) and is ergodic, then its unique invariant measure is mixing.

Since spin flip systems, as defied in Chapter III of [12], are special cases of these processes for which (3.3) follows from (3.1), this answers affirmatively the question raised in Problem 7 of that chapter.

For certain interacting particle systems, it is easier to show the convergence of $\mu S(t)$ if one assumes that $\mu \in \mathcal{S}$. In these cases part (i) of Theorem (1.4)

may be applicable. In particular if $\mathscr{I} \cap \mathscr{S} = \{\nu\}$ and $\mu S(t)$ converges for all $\mu \in \mathscr{S}$, then ν must be ergodic.

4. Systems of stochastic differential equations

In this section we apply part (i) of Theorem (1.4) to the processes studied by Holley and Stroock in [11]. To do so we need the following proposition which will be used in the next section too. In the statement of the proposition $\mathscr{A}$ is a subset of $\mathscr{M}$ and X and $\mathscr{M}$ are as in the introduction. Although we do not require $\mathscr{A}$ to be convex, we will only apply the proposition in situations in which it is.

Proposition (4.1). *Suppose X and $\mathscr{A}$ are compact. Let λ be a Borel probability measure on $\mathscr{M}$, ν an extreme point in $\mathscr{A}$ and $K_n : \mathscr{M} \to \mathscr{M}$ a sequence of measurable maps such that $\int K_n(\mu) \, d\lambda(\mu) = \nu$ for all n. If for λ-almost all μ all weak limits of $K_n(\mu)$ belong to $\mathscr{A}$, then there exists a sequence $n_k \uparrow \infty$ satisfying $\lim_k K_{n_k}(\mu) = \nu$ for λ-almost all μ.*

Proof. In this proof statements about convergence in probability and almost sure convergence are related to the probability space $(\mathscr{M}, \lambda)$.

Let λ_n be the measure induced by λ under the mapping K_n. The assumptions of the proposition immediately imply that $\int \mu \, d\lambda_n = \nu$ and that all weak limits of λ_n concentrate on $\mathscr{A}$. Therefore, if γ is a weak limit of λ_n, then $\int \mu \, d\gamma = \nu$. Since ν is extreme, $\gamma = \delta_\nu$. Therefore $\lim_n \lambda_n = \delta_\nu$ (weakly). This implies that K_n converges to the constant ν in probability. Hence, there exists a subsequence of K_n converging to ν almost surely.

In the final part of this section we will assume the reader to be familiar with the contents of [11]. The processes studied in §§4 and 5 of that paper have as state space $T^{\mathbf{Z}^d}$, where $T = \{z \in \mathbf{C} : |z| = 1\}$. It follows from the construction of these processes, more precisely from Lemma (1.2) and Theorem (1.3) of [11], that for all pairs of continuous real-valued functions f and g, depending on finitely many coordinates, we have

$$\lim_{|n_1| \to \infty} \|S(t)(f T^{n_1} g) - S(t) f S(t)(T^{n_1} g)\| = 0.$$

Hence (1.5) is satisfied for all $\nu \in \mathscr{M}$. The fact that $S(t)$ commutes with the translations of $\mathbf{Z}^d$ follows also from the construction, as explained in [11]. Finally the hypothesis for part (i) of Theorem (1.4) is a consequence of Proposition (4.1) in this paper and Theorem (4.35) of [11]. Therefore, for the processes studied in §4 of [11] we conclude that *all elements of $(\mathscr{G} \cap \mathscr{S})_e$ are ergodic* where $\mathscr{G}$ is the set of Gibbs states.

Remark. As explained at the beginning of §4 of [11] all Gibbs states are reversible for the process. Therefore $\mathscr{G} \subset \mathscr{I}$. Corollary (4.37) of [11] states that $\mathscr{I} \cap \mathscr{S} \subset \mathscr{G}$. Hence $(\mathscr{I} \cap \mathscr{S})_e = (\mathscr{G} \cap \mathscr{S})_e$.

5. STOCHASTIC ISING MODELS, GIBBS STATES

In this section we apply Theorem (1.4) to stochastic Ising models, as defined in Chapter IV of [12]. We will assume that the reader is familiar with the notation of that chapter.

Given a translation invariant potential $\{J_R\}$ on $\mathbf{Z}^d$, such that

$$(5.1) \qquad\qquad \sum_{R \ni 0} |R|\, |J_R| < \infty,$$

the corresponding stochastic Ising model satisfies the hypothesis of Theorem (4.6) of Chapter I of [12]. The conclusion of that theorem shows that (1.5) is satisfied for all cylinder functions f and g and all $\nu \in \mathcal{M}$. To see that the hypothesis of part i) of Theorem (1.4) is fulfilled too, we appeal to Proposition (4.1) and to the fact that all weak limits of $\mu S(t)$ $(\mu \in \mathscr{S})$ belong to $\mathscr{G}$. This last statement was proved by Holley [10] for finite range potentials, and extended to potentials satisfying (5.1) by Higuchi and Shiga [9]. These considerations show that *all elements of* $(\mathscr{G} \cap \mathscr{S})_e$ *are ergodic* where $\mathscr{G}$ denotes the set of Gibbs states.

We will show in §7 that elements of $(\mathscr{G} \cap \mathscr{S})_e$ may not be mixing. We do, however, have the following result: *all elements of* $\mathscr{G}_e \cap \mathscr{S}$ *are mixing.* To prove this, take a countable set of cylinder functions $\{f_m\}$, dense in $C(X)$, then apply Theorem (4.13) in Chapter IV of [12] and the diagonal argument, to obtain a sequence t_n diverging to infinity and satisfying

$$\lim_n S(t_n) f_m(\eta) = \int f_m \, d\nu \quad \nu\text{-a.e.}\ \forall m.$$

Then apply part (ii) of Theorem (1.4).

Remarks. As in the previous section $\mathscr{G} \cap \mathscr{S} = \mathscr{I} \cap \mathscr{S}$. This follows from the results of Holley, extended by Higuchi and Shiga, which were mentioned above.

The second result in this section is not new. A direct proof, not recurring to stochastic Ising models, can be found in [14, p. 20]. Better results are known when Dobrushin's uniqueness condition is satisfied. This condition implies that $\mathscr{G}$ is a singleton, and that its unique element has exponentially decaying correlations (see [2 and 3]). For a more recent treatment of the decay of the correlations under that condition, we refer the reader to [7]. The reader is also referred to [4], where for each finite $V \subset \mathbf{Z}^d$, a condition C_V is given. It is then proved there, that if C_V is satisfied for some V, then there is a unique Gibbs state and its correlations decay exponentially. This generalizes the above results, because C_V coincides with the previous condition when V is a singleton.

6. LINEAR SYSTEMS ON $[0, \infty)^{\mathbf{Z}^d}$

These processes evolve on appropriate subspaces of $[0, \infty)^S$, for some countable set S. They are called linear because given one η_t, there exists a family $\{A_t(x, y),\ t \geq 0,\ x, y \in S\}$ of nonnegative random matrices such that

$\eta_t(x) = \sum_{y \in S} A_t(x, y)\eta_0(y)$. The main reference to these systems is Chapter IX of [12] and we assume that the reader is familiar with, at least, the first section of that chapter. Besides the assumptions made in that section, we will only consider cases in which $S = \mathbf{Z}^d$ and the formal generator of the process is invariant under translations on $\mathbf{Z}^d$. Under these hypotheses it is shown in §2 of Chapter IX of [12] that $\delta_\beta S(t)$ converges, as t goes to infinity (β denoting the configuration $\eta(x) \equiv \beta$). The limit is called ν_β, belongs to $\mathscr{S}$ and, as proved in [12], either it is δ_0 or it satisfies $\int \eta(x)\,d\nu_\beta = \beta$. In this section we prove that ν_β *is ergodic*, thus solving Problem 4 on page 468 of [12]. Since the case in which $\nu_\beta = \delta_0$ is trivial, we assume from now on that $\int \eta(x)\,d\nu_\beta = \beta > 0$.

Given one of the above processes, the subspace X of $[0, \infty)^{\mathbf{Z}^d}$ on which it is defined is determined in [12], in the following way: Lemma 1.6 on page 423 of [12] constructs a function $\alpha \colon \mathbf{Z}^d \to \mathbf{R}_+$, then X is defined as

$$\left\{ \eta \in [0, \infty)^{\mathbf{Z}^d} \colon \sum_{x \in \mathbf{Z}^d} \eta(x)\alpha(x) < \infty \right\}.$$

The topology of X is now determined by means of the following "norm":

$$\|\eta - \xi\| = \sum_{x \in \mathbf{Z}^d} |\eta(x) - \xi(x)|\alpha(x).$$

In this way, X becomes a complete separable metric space.

Let $\mathscr{L}$ be the subset of $C(X)$ formed by the functions f for which there exists a constant L such that $|f(\eta) - f(\xi)| \le L\|\eta - \xi\| \ \forall \eta, \xi \in X$. Given $f \in \mathscr{L}$, call $L(f)$ the smallest constant satisfying this inequality. Fix a sequence of finite subsets S_n of $\mathbf{Z}^d$, such that $S_n \uparrow \mathbf{Z}^d$, and call $\mathscr{L}_n$ the subset of elements of $\mathscr{L}$ not depending on coordinates off S_n.

Remark. Here we have departed a little from the notation of [12]. There, elements of $\mathscr{L}$ have continuous partial derivatives and may be unbounded.

There are, however, no obstacles to the construction of a semigroup $S(t)$ on $\mathscr{L}$, as it is done in [12]. We make now five assertions:

(1) By appropriately choosing the function α, X can be made to be invariant under translations.

(2) The semigroup $S(t)$ can be extended to $C(X)$ and the extension is a Feller semigroup as defined in the introduction.

(3) The extended semigroup commutes with translations on $\mathbf{Z}^d$.

(4) If $\nu \in \mathscr{S}$ and $B = \bigcup_{n=1}^\infty \mathscr{L}_n$ then (1.5) is satisfied.

(5) B is dense in $L^2(\nu)$.

The proofs of the first four assertions depend heavily on the construction of the process and are quite unrelated to the rest of this paper. These proofs do not differ substantially from proofs of similar assertions that have been made for other processes. The reader should, however, note that a result as strong as

Theorem (4.6) in Chapter I of [12], does not hold in the present context. Nevertheless, the proof of the fourth assertion can be done by means of standard coupling techniques and the use of some of the ideas appearing in the construction of these linear systems. The proof of the last assertion is standard. For these reasons, all these proofs will be omitted.

Since X is a complete separable metric space and $\nu_\beta \in \mathscr{S}$, there is a unique ergodic decomposition

$$\nu_\beta = \int \mu \, d\lambda(\mu).$$

The ergodicity of ν_β follows now from Lemma (6.1) below and part (i) of Theorem (1.4). The proof of Lemma (6.1) is an adaptation of the proof of Lemma 2.9 on page 436 of [12].

Lemma (6.1). *There exists a sequence $t_n \uparrow \infty$ such that $\int f \, d\mu S(t_n)$ converges to $\int f \, d\nu_\beta$ for all $f \in B$ and λ-almost all μ.*

Proof. For $\mu \in \mathscr{S}$, define $\varphi(\mu) = \int \eta(x) \, d\mu$. In view of our assumption concerning ν_β we have

$$\int \varphi(\mu) \, d\lambda(\mu) = \beta.$$

Let ξ be an element of $[0, \infty)^{\mathbf{Z}^d}$, different from 0 and with finite support. From Jensen's inequality we obtain

$$\exp\left[-\varphi(\mu) \sum \xi(x)\right] \le \int \exp\left[-\sum \eta(x)\xi(x)\right] d\mu.$$

(In this proof all sums are over $\mathbf{Z}^d$ and the integrated variable is η, unless it is otherwise specified.) Let ξ_t be the dual process of η_t (see pages 430–431 of [12]). Then

$$E^\xi\left[\exp\left(-\varphi(\mu) \sum \xi_t(x)\right)\right] \le \int E^\xi\left[\exp\left(-\sum \eta(x)\xi_t(x)\right)\right] d\mu.$$

By Theorem 1.25 on page 431 of [12], this last inequality can be written as

$$(6.2) \qquad \int \exp\left[-\frac{\varphi(\mu)}{\beta} \sum \eta(x)\xi(x)\right] d\delta_\beta S(t)$$
$$\le \int \exp\left[-\sum \eta(x)\xi(x)\right] d\mu S(t).$$

For $\gamma \ge 0$ define

$$\Phi_\xi(\gamma) = \int \exp\left[-\gamma \sum \eta(x)\xi(x)\right] d\nu_\beta.$$

Then

$$\Phi_\xi\left(\frac{\varphi(\mu)}{\beta}\right) - \int \exp\left[-\frac{\varphi(\mu)}{\beta} \sum \eta(x)\xi(x)\right] \delta_\beta S(t) \to 0 \quad \text{as } t \to \infty.$$

Therefore, by (6.2)

$$(6.3) \qquad \Phi_\xi\left(\frac{\varphi(\mu)}{\beta}\right) \le \int \exp\left[-\sum \eta(x)\xi(x)\right] d\mu S(t) + f(\mu, t)$$

where

(6.4) $|f(\mu, t)| \le 1 \quad \forall \mu, t, \quad$ and $\quad \lim_t f(\mu, t) = 0 \quad \forall \mu.$

Hence

$$\int \Phi_\xi \left(\frac{\varphi(\mu)}{\beta} \right) d\lambda(\mu) \le \Phi_\xi(1) + \int f(\mu, t) d\lambda(\mu).$$

Letting $t \to \infty$, it follows from (6.4) and the Dominated Convergence Theorem that

$$\int \Phi_\xi \left(\frac{\varphi(\mu)}{\beta} \right) d\lambda(\mu) \le \Phi_\xi(1).$$

Since Φ_ξ is strictly convex, this implies that

(6.5) $\varphi(\mu) = \beta \quad \lambda\text{-a.e.}$

Integrate (6.3) with respect to λ; in view of (6.5) this yields

(6.6)
$$\Phi_\xi(1) \le \int \left[\int \exp \left[-\sum \eta(x)\xi(x) \right] d\mu S(t) + f(\mu, t) \right] d\lambda(\mu)$$
$$= \Phi_\xi(1) + \int f(\mu, t) d\lambda(\mu).$$

By (6.4) and the Dominated Convergence Theorem the limit as $t \to \infty$ of the right-hand side above is $\Phi_\xi(1)$. From (6.3) and (6.5) we obtain

$$\Phi_\xi(1) \le \int \exp \left[-\sum \eta(x)\xi(x) \right] d\mu S(t) + f(\mu, t) \quad \lambda\text{-a.e.}$$

Hence, there exists $h(\mu, t) \ge 0$ such that

(6.7) $f(\mu, t) + \int \exp \left[-\sum \eta(x)\xi(x) \right] d\mu S(t) = \Phi_\xi(1) + h(\mu, t) \quad \lambda\text{-a.e.}$

Plugging this in (6.6) we get

$$\Phi_\xi(1) \le \Phi_\xi(1) + \int h(\mu, t) d\lambda(\mu) \le \Phi_\xi(1) + \int f(\mu, t) d\lambda(\mu).$$

In view of the observation following (6.6), this implies that $h(\mu, t) \to 0$ in $L^1(\lambda)$. Hence, there exists a sequence $t_n \uparrow \infty$, such that

(6.8) $h(\mu, t_n) \to 0 \quad \lambda\text{-a.e.}$

It follows from (6.4), (6.7) and (6.8) that

(6.9) $\lim_{n \to \infty} \int \exp \left[-\sum \eta(x)\xi(x) \right] d\mu S(t_n)$
$$= \Phi_\xi(1) = \int \exp \left[-\sum \eta(x)\xi(x) \right] d\nu_\beta \quad \lambda\text{-a.e.}$$

Using the diagonal argument we obtain a further subsequence, also called t_n, for which (6.9) holds for all $\xi \ne 0$ with rational coordinates and finite support. By monotonicity and continuity, (6.9) must then hold for all ξ with finite support. Therefore, for λ-almost all μ the finite-dimensional Laplace transforms

of $\mu S(t_n)$ converge to the corresponding finite-dimensional Laplace transforms of ν_β. This proves the lemma.

Remark. In [12], α is, at a certain point, assumed to be symmetric. Under that assumption, the process η_t and its dual ξ_t have the same state space. Here, that assumption is not necessary, because the dual process is only needed on the set $\{\xi\colon \sum_{x\in\mathbf{Z}^d}\xi(x) < \infty\}$.

7. Counterexamples and open problems

We start this section with examples showing what can happen if the hypotheses of Theorem (1.4) are relaxed. We will not give complete proofs of our statements, but the interested reader will find no serious difficulty in providing them. We start with an example of a translation invariant process, satisfying (1.5) and possessing a unique invariant measure which is not ergodic.

Let $X = T^{\mathbf{Z}}$ where $T = \{z \in \mathbf{C}\colon |z| = 1\}$. Consider a process on X with the following dynamics:

(a) All coordinates of $\eta \in X$ turn counterclockwise at the same constant speed $v > 0$.

(b) At times given by independent Poisson processes of parameter one, coordinates choose a neighbor independently of the Poisson processes. Each neighbor has probability $\frac{1}{2}$ of being chosen. Then the coordinate becomes equal to the chosen neighbor.

It is easy to verify that this process and the one-dimensional nearest neighbor voter model analyzed in Chapter V of [12], form blocs of equal coordinates in the same way. Then, by Corollary 1.13 on page 231 of [12], an invariant measure for any of these processes must concentrate on constant configurations. From this it is easy to deduce that the process we are considering has a unique invariant measure ν. This measure is the only one concentrating on constant configurations and having as marginals the normalized Lebesgue measure on T. Of course, this measure is not ergodic.

For an example satisfying the hypothesis of part (i) of Theorem (1.4) but not satisfying the conclusion of part (ii) of that theorem, we refer the reader to [13]. It follows from the results in that paper that the nearest neighbor antivoter model in one dimension has as extremal invariant measures the point masses on

$$\eta(x) = \begin{cases} 1 & \text{if } x \text{ is even}, \\ 0 & \text{if } x \text{ is odd}, \end{cases}$$

and on

$$\xi(x) = \begin{cases} 0 & \text{if } x \text{ is even}, \\ 1 & \text{if } x \text{ is odd}. \end{cases}$$

Hence the unique element of $\mathscr{I}\cap\mathscr{S}$ is $\nu = \frac{1}{2}(\delta_\eta + \delta_\xi)$. It is also proved in [13] that $\lim_t \mu S(t) = \nu$ for all $\mu \in \mathscr{S}$. Obviously ν is ergodic but not mixing.

Another example of this kind, for which all the transition rates are strictly positive, is given by a nonattractive two-dimensional Ising model. Take as

potential $J_K = \beta$ if $K = \{x, y\}$ with $\|x - y\| = 1$ (Euclidean norm) and $J_K = 0$ otherwise. For $\beta < 0$ and $|\beta|$ sufficiently large, $\mathscr{G} \cap \mathscr{S}$ is a singleton. Its unique element is ergodic as proved in §5, but it is not mixing. This last assertion follows from the facts stated in the second paragraph on page 204 of [12] and the results of [1 and 8].

For an example of a translation invariant, ergodic process in $\{0, 1\}^{\mathbf{Z}}$, whose unique invariant measure is not ergodic we refer the reader to page 178 of [12] (the example given in Problem 7). Of course, this example does not satisfy (1.5).

Theorem (1.4) and the first two examples above motivate two questions we have not been able to answer. In what follows we only consider translation invariant interacting particle systems on $\{0, 1\}^{\mathbf{Z}^d}$, for which (3.1) and (3.2) hold. Under these hypotheses *is it the case that $(\mathscr{I} \cap \mathscr{S})_e \subset \mathscr{S}_e$?* Note that the first example given in this section has an important deterministic component. Since particle systems as above do not share that property, it seems plausible that for all $\mu \in \mathscr{S}$, $\mu S(t)$ converges. This, together with part (i) of Theorem (1.4) will imply that the answer to the above question is affirmative.

The second question is motivated by the results of §5. *Is it the case that elements of $\mathscr{I}_e \cap \mathscr{S}$ are mixing?*

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